A Resilience Indicator for Killed Diffusion Asset Price Models

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ABSTRACT

We formalise a concept of resilience of asset price models which follow killed Markov diffusion processes over the nonnegative real line by identifying a useful reduction of the conditional probability of reaching breakeven and avoiding ruin after a large drawdown. This indicator is, in particular, a risk management tool for purely long positions on possibly defaultable assets. We exemplify clearly this quantity for geometric Brownian motion and the Bessel process of possibly non-integer dimension. We only use facts that can be found in or derived from foundational references of stochastic analysis: [2] and [1] in general, and [3] for the Bessel process in particular.

Let $X = (X_t)_{t\geq 0}$ be a linear strong Markov diffusion on $I = [0, \infty)$ with probability measure P^x associated to X started at point $x \in I$. We assume its infinitesimal operator has the second order differential form

$$\mathscr{G} f(x) = \frac{1}{2} a^2(x) \frac{d^2}{dx^2} f(x) + b(x) \frac{d}{dx} f(x)$$

and we suppose that a, b are nice enough such that there exists a unique weak solution to the stochastic differential equation

$$dX_t = b(X_t)dt + a(X_t)dW_t,\, X_0 = x,\, t < \zeta$$

where $\zeta := \inf\{t \geq 0 : X_t \in \{0, \infty\}\}$ with convention $\inf \emptyset = \infty$ (sufficient conditions on a, b are the Engelbert–Schmidt conditions, see e.g. ([2]; Th.5.7 p. 335)). The weak solution X represents our asset price model with continuous paths up to ζ; this is a *killed diffusion*. The diffusion scale function $s(x)$ associated to X solves the ODE $\mathscr{G} s(x) = 0$ and has the form

$$s(x) = \int_{\varepsilon_1}^{x} \exp\left(-2 \int_{\varepsilon_2}^{y} \frac{b(z)}{a^2(z)} dz \right) dy$$

for some $\varepsilon_1, \varepsilon_2 > 0$. By applying Itô's formula to $s(X_t), t < \zeta$ we obtain that $ds(X_t) = a(X_t)s'(X_t)dW_t, t < \zeta$, which is at least a local martingale. Let $\alpha \in (0, 1), x > 0$ and

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define the stopping times

$$\tau(\alpha) := \inf\{t \geq 0 : X_t = \alpha x\}$$
$$\sigma(\alpha) := \inf\{t \geq \tau(\alpha) : X_t = x\}$$

which are respectively, the first time of a large drawdown (for $\alpha \ll 1$) and the corresponding first time of recovery. We define the *resilience* of X as the probability that $\sigma(\alpha) < \zeta$ conditional on $\tau(\alpha) < \infty$. Note that $\{\sigma(\alpha) < \zeta\} = \{\sigma(\alpha) < \zeta\} \cap \{\tau(\alpha) < \infty\}$. We obtain, by the strong Markov property of X,

$$\begin{aligned} P^x(\sigma(\alpha) < \zeta | \tau(\alpha) < \infty) &:= \frac{P^x(\sigma(\alpha) < \zeta)}{P^x(\tau(\alpha) < \infty)} \\ &= \frac{E^x[P^x(\sigma(\alpha) < \zeta | \mathscr{F}_{\tau(\alpha)})\mathbf{1}_{\{\tau(\alpha)<\infty\}}]}{P^x(\tau(\alpha) < \infty)} \\ &= \frac{P^{\alpha x}(\sigma(\alpha) < \zeta) P^x(\tau(\alpha) < \infty)}{P^x(\tau(\alpha) < \infty)} \\ &= P^{\alpha x}(\sigma(\alpha) < \zeta) \end{aligned}$$

so the resilience of X simplifies to the probability of reaching x while starting from αx, while also avoiding ruin. Since $x > \alpha x$ and the paths of X are continuous, we also have $P^{\alpha x}(\sigma(\alpha) < \zeta) = P^{\alpha x}(\sigma(\alpha) < \tau_0)$ where $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$, the ruin time of X. Therefore we conclude:

$$\boxed{P^x(\sigma(\alpha) < \zeta | \tau(\alpha) < \infty) = P^{\alpha x}(\sigma(\alpha) < \tau_0)}$$

In general, if $\tau_\varepsilon := \inf t \geq 0 : X_t = \varepsilon$, then (see e.g. ([1]; §4 pp. 13–14)) $P^{x\alpha}(\sigma(\alpha) < \tau_\varepsilon) = (s(\alpha x) - s(\varepsilon))/(s(x) - s(\varepsilon))$. It follows that $P^{x\alpha}(\sigma(\alpha) < \tau_\varepsilon) \to P^{\alpha x}(\sigma(\alpha) < \tau_0)$ as $\varepsilon \downarrow 0$. Therefore, the general expression for the resilience of X is

$$P^x(\sigma(\alpha) < \zeta | \tau(\alpha) < \infty) = \begin{cases} 1 & \lim_{\varepsilon \downarrow 0} s(\varepsilon) = -\infty \\ \frac{s(\alpha x) - s(0^+)}{s(x) - s(0^+)} & \lim_{\varepsilon \downarrow 0} s(\varepsilon) > -\infty \end{cases}$$

We shall now illustrate and directly prove two straightforward examples.

(1) If $b(x) = \mu x, \mu \in \mathbb{R}$ and $a(x) = \gamma x, \gamma > 0$ then X is a geometric Brownian motion. In particular, $P^z(\zeta = \infty) = 1$ for all $z > 0$. We can choose

$$s(x) = \begin{cases} \frac{x^{\beta^*} - 1}{\beta^*} & \beta^* \neq 0 \\ \ln x & \beta^* = 0 \end{cases}$$

where $\beta^* := 1 - 2\mu/\gamma^2$. In particular, it is straightforward to check that $s(X_t)$ is a true martingale by taking conditional expectations, therefore also $X_t^{\beta^*}$ is a true martingale. We get by optional stopping theorem

$$(\alpha x)^{\beta^*} = E^{\alpha x}[X_{\sigma(\alpha) \wedge t}^{\beta^*}] = x^{\beta^*} P^{\alpha x}(\sigma(\alpha) \leq t) + E^{\alpha x}[X_t^{\beta^*} \mathbf{1}_{\{\sigma(\alpha) > t\}}]$$

We have $X_t \leq x$ on $\{\sigma(\alpha) > t\}$. If $\mu < \gamma^2/2$ then $\beta^* > 0$ and so $X_t^{\beta^*} \leq x^{\beta^*}$, while

$X_t \to 0$ a.s. So $P^{x\alpha}(\sigma(\alpha) < \infty) = \alpha^{\beta^*}$. If $\mu \geq \gamma^2/2$ then $P^{x\alpha}(\sigma(\alpha) < \infty) = 1$, by a standard argument; indeed $\ln X_t \to \infty$ if $\beta^* < 0$, while $\ln X_t$ is point recurrent over $\mathbb{R}$ if $\beta^* = 0$. Therefore we conclude:

$$P^x(\sigma(\alpha) < \zeta | \tau(\alpha) < \infty) = \begin{cases} 1 & \mu \geq \gamma^2/2 \\ \alpha^{1-2\mu/\gamma^2} & \mu < \gamma^2/2 \end{cases}$$

(2) If $b(x) = c/x, c > 0$ and $a(x) \equiv 1$ then X is a Bessel process of dimension $2c+1$. If $c = 1/2$, then the dimension is exactly 2, so X is recurrent over $(0, \infty)$. If $c > 1/2$, then X is strictly positive and $X_t \to \infty$ P^z-a.s. for any $z > 0$. On the other hand, if $c \in (0, 1/2)$ then $P^z(\tau_0 < \infty) > 0$ and $X_t \to 0$ P^z-a.s. for any $z > 0$. For all values of $c > 0$, we have that ∞ is inaccessible in finite time, so $\zeta = \tau_0$. We can choose

$$s(x) = \begin{cases} \frac{x^{1-2c}-1}{1-2c} & c \neq 1/2 \\ \ln x & c = 1/2 \end{cases}$$

For $c < 1/2$, we can see by Itô's formula that under $P^{\alpha x}$ we obtain

$$s(X_{\sigma(\alpha)\wedge\zeta\wedge t}) = s(\alpha x) + \int_0^{\sigma(\alpha)\wedge\zeta\wedge t} X_s^{-2c} dW_s$$

Since $X_{\sigma(\alpha)\wedge\zeta\wedge t} \in [0, x]$, $s(X_{\sigma(\alpha)\wedge\zeta\wedge t})$ is a true (bounded) martingale, and so is $(X_{\sigma(\alpha)\wedge\zeta\wedge t})^{1-2c}$. We obtain

$$\begin{aligned}(\alpha x)^{1-2c} &= E^{\alpha x}[(X_{\sigma(\alpha)\wedge\zeta\wedge t})^{1-2c}] \\ &= x^{1-2c} P^{\alpha x}(\sigma(\alpha) \leq \zeta \wedge t) + E^{\alpha x}[(X_{\zeta\wedge t})^{1-2c} \mathbf{1}_{\{\zeta\wedge t < \sigma(\alpha)\}}]\end{aligned}$$

Since $c < 1/2$, we have $(X_{\zeta\wedge t})^{1-2c} \leq x^{1-2c}$ on $\{\zeta \wedge t < \sigma(\alpha)\}$. By dominated convergence and the fact that $X_t \to 0$ (in fact, a.s. in finite time; see e.g. ([3]; (proof of) Prop. 2.1 pp. 7–8)), $P^{\alpha x}$-a.s. if $c < 1/2$, we obtain $P^{\alpha x}(\sigma(\alpha) < \zeta) = \alpha^{1-2c}$ hence:

$$P^x(\sigma(\alpha) < \zeta | \tau(\alpha) < \infty) = \begin{cases} 1 & c \geq 1/2 \\ \alpha^{1-2c} & c \in (0, 1/2) \end{cases}$$

because if $c \geq 1/2$ then $P^{\alpha x}(\tau_0 = \infty) = 1$ and X is either recurrent or $X_t \to \infty$ $P^{\alpha x}$-a.s., and has continuous paths.

References

[1] Borodin, A. N. and Salminen, P. (2002). *Handbook of Brownian Motion - Facts and Formulae, Second Edition*. Birkhäuser Basel.

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[3] Lawler, G. F. (2018). *Notes on the Bessel process*. https://api.semanticscholar.org/CorpusID:52200396