On Delta-hedging of leveraged derivatives strategies

Pietro Maria Sparago[a]

[a]Department of Statistics, London School of Economics, London, UK

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ABSTRACT
In this note we discuss two leveraged derivatives strategies in continuous time, both in the Black-Scholes framework and in its immediate, standard extension where implied volatility differs from realised volatility: (i) a rolling leverage strategy in which the amount of derivatives units on leverage is held constant in time, and (ii) a LETF-like portfolio constructed through a straightforward generalisation of the method of [1], in which the relative weight of leveraged derivatives is held constant in time. We focus on the closed-form of the Delta-hedging weight in both cases, and the consequences on the Gamma exposure under the implied $\neq$ realised volatility scenario.

1. Rolling leverage strategy in continuous time

Let $S = (S_t)_{t\geq 0}$ be the underlying asset price and $V(t, S_t)$ the time-t price of a derivative on S. Suppose that, in discrete time from t to $t + \Delta t$, the leveraged strategy longs one unit of V and borrows γ units at rate r at time t, repaying the corresponding amount at time $t + \Delta t$. The cash flow profile is:

$$\begin{array}{cccc} & \text{Long one unit} & \text{Levered } \gamma \text{ units} & \text{Leverage} \\ t: & -V(t,S_t) & -\gamma V(t,S_t) & +\gamma V(t,S_t) \\ t+\Delta t: & V(t+\Delta t, S_{t+\Delta t}) & \gamma V(t+\Delta t, S_{t+\Delta t}) & -\gamma e^{r\Delta t} V(t,S_t) \end{array}$$

The profit-and-losses (PnL) differential is then

$$\Delta\Pi_{t\to t+\Delta t} = (1+\gamma)(V(t+\Delta t, S_{t+\Delta t}) - V(t,S_t)) - \gamma V(t,S_t)(e^{r\Delta t} - 1)$$

If we consider the Black-Scholes model for S (i.e. geometric Brownian motion dynamics), then the corresponding stochastic differential as $\Delta t \downarrow 0$ is

$$d\Pi_t = (1+\gamma) dV(t,S_t) - r\gamma V(t,S_t)dt$$

since $\lim_{\Delta t\downarrow 0}(e^{r\Delta t} - 1)/\Delta t = r$. The total PnL from time t to time $T > t$ is then

$$\Pi_T - \Pi_t = (1+\gamma)(V(T,S_T) - V(t,S_t)) - r\gamma \int_t^T V(u,S_u)du$$

CONTACT P. M. Sparago. Email: p.sparago@lse.ac.uk

Under the Black-Scholes risk-neutral measure Q, we obtain, as expected,

$$\begin{aligned}
E_t^Q[\Pi_T - \Pi_t] &= (1+\gamma)(E_t^Q[V(T,S_T)] - V(t,S_t)) - r\gamma \int_t^T E_t^Q[V(u,S_u)]du \\
&= (1+\gamma)V(t,S_t)(e^{r(T-t)} - 1) - r\gamma V(t,S_t)\int_t^T e^{r(u-t)}du \\
&= V(t,S_t)(e^{r(T-t)} - 1)
\end{aligned}$$

since $(e^{-rt}V(t,S_t))_{t\geq 0}$ is a Q-martingale. Now consider the model relaxation where the Black-Scholes implied volatility of V is different from the realised volatility, resp. $\sigma_I \neq \sigma_R$. The derivative price $(t,s) \mapsto V(t,s)$ solves the Black-Scholes PDE on $(0,T)\times(0,\infty)$

$$\frac{\partial V}{\partial t}(t,s) + rs\frac{\partial V}{\partial s}(t,s) + \frac{1}{2}s^2\sigma_I^2\frac{\partial^2 V}{\partial s^2}(t,s) = rV(t,s)$$

while the underlying P-dynamics are $dS_t = \mu S_t dt + \sigma_R S_t dW_t$. The PnL stochastic differential has the representation:

$$\begin{aligned}
d\Pi_t &= (1+\gamma)dV(t,S_t) - r\gamma V(t,S_t)dt \\
&= (1+\gamma)\left(\frac{\partial V}{\partial t}(t,S_t) + \mu S_t\frac{\partial V}{\partial s}(t,S_t) + \frac{1}{2}S_t^2\sigma_R^2\frac{\partial^2 V}{\partial s^2}(t,S_t)\right)dt \\
&- r\gamma V(t,S_t)dt + (1+\gamma)\frac{\partial V}{\partial s}(t,S_t)\sigma_R S_t dW_t \\
&= (1+\gamma)\left((\mu - r)S_t\frac{\partial V}{\partial s}(t,S_t) + \frac{1}{2}S_t^2(\sigma_R^2 - \sigma_I^2)\frac{\partial^2 V}{\partial s^2}(t,S_t)\right)dt \\
&+ rV(t,S_t)dt + (1+\gamma)\frac{\partial V}{\partial s}(t,S_t)\sigma_R S_t dW_t
\end{aligned}$$

It is then clear that the Delta-hedge is

$$\boxed{\delta_t := (1+\gamma)\frac{\partial V}{\partial s}(t,S_t)}$$

Indeed, consider the Delta-hedged PnL differential $d\tilde{\Pi}_t$ where the short proceedings are reinvested at rate r. We obtain, after simplification,

$$\begin{aligned}
d\tilde{\Pi}_t &= d\Pi_t - \delta_t dS_t + r\delta_t S_t dt \\
&= d\Pi_t - (1+\gamma)\frac{\partial V}{\partial s}(t,S_t)dS_t + r(1+\gamma)S_t\frac{\partial V}{\partial s}(t,S_t)dt \\
&= rV(t,S_t)dt + \frac{1+\gamma}{2}(\sigma_R^2 - \sigma_I^2)S_t^2\frac{\partial^2 V}{\partial s^2}(t,S_t)dt
\end{aligned}$$

Hence, we conclude that Delta-hedging the rolling leverage strategy scales the dollar Gamma exposure $S_t^2\frac{\partial^2 V}{\partial s^2}(t,S_t)$ by the leverage multiplier $1+\gamma$, under the above model relaxation. In the case $\sigma_I = \sigma_R$, we recover the Black-Scholes case, which imposes the risk-free return $rV(t,S_t)dt$ on the Delta-hedged PnL.

2. Leveraged ETF-like derivatives portfolio

We now follow the construction of [1], but for a derivatives LETF (leveraged exchange-traded fund). Assuming $(V(t,S_t))_{t\geq 0}$ is strictly positive and $\beta > 1$, the ("bullish") LETF portfolio $L = (L_t)_{t\geq 0}$ with an initial endowment $L_0 > 0$ is constructed by going long on $\beta L_t/V(t,S_t)$ units of $V(t,S_t)$ while borrowing $(\beta-1)L_t$ at the rate r, and then rebalancing accordingly. The stochastic differential of the portfolio value is then

$$dL_t = \frac{\beta L_t}{V(t,S_t)} dV(t,S_t) - r(\beta-1)L_t dt$$

If we consider the Black-Scholes model for S, then we obtain

$$\frac{dV(t,S_t)}{V(t,S_t)} = rdt + \sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dW_t^Q$$

By applying Itô's formula to $\ln L_t$ and $\ln V(t,S_t)$,

$$\begin{aligned}
d\ln V(t,S_t) &= \left(r - \frac{1}{2}\left(\sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t)\right)^2\right)dt + \sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dW_t^Q \\
d\ln L_t &= \left(r - \frac{\beta^2}{2}\left(\sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t)\right)^2\right)dt + \beta\sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dW_t^Q \\
&= \left((1-\beta)r - \frac{\beta(\beta-1)}{2}\left(\sigma \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t)\right)^2\right)dt + \beta d\ln V(t,S_t)
\end{aligned}$$

Hence the solution for the SDE of L is found in terms of $V(t,S_t)$: for any $t < T$

$$\frac{L_T}{L_t} = \left(\frac{V(T,S_T)}{V(t,S_t)}\right)^\beta \exp\left((1-\beta)r(T-t) - \frac{\beta(\beta-1)}{2}\int_t^T \left(\sigma \frac{S_u}{V(u,S_u)} \frac{\partial V}{\partial s}(u,S_u)\right)^2 du\right)$$

similarly as in ([1]; Eq. 10 p. 592). By changing measure from Q to P we obtain:

$$\begin{aligned}
dL_t &= rL_t dt + \sigma\beta L_t \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dW_t^Q \\
&= L_t\left(r + (\mu - r)\beta \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t)\right)dt + \sigma\beta L_t \frac{S_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dW_t
\end{aligned}$$

The Delta-hedge is then

$$\boxed{\delta_t = \beta L_t \frac{1}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t)}$$

which implies (with short proceedings at rate r)

$$
\begin{aligned}
&dL_t - \delta_t dS_t + r\delta_t S_t dt \\
&= dL_t - \beta L_t \frac{1}{V(t,S_t)} \frac{\partial V}{\partial Ss}(t,S_t) dS_t + rS_t \beta L_t \left(\frac{1}{V(t,S_t)} S_t \frac{\partial V}{\partial s}(t,S_t) \right) dt \\
&= \frac{L_t}{V(t,S_t)} \left(rV(t,S_t) \right) dt = rL_t dt
\end{aligned}
$$

Now consider the Black-Scholes model relaxation of the previous section, so that $\sigma_I \neq \sigma_R$. We get

$$
\begin{aligned}
dL_t &= \beta \frac{L_t}{V(t,S_t)} \left(\frac{\partial V}{\partial t}(t,S_t) + \frac{1}{2}\sigma_R^2 S_t^2 \frac{\partial^2 V}{\partial s^2}(t,S_t) + \mu S_t \frac{\partial V}{\partial s}(t,S_t) \right) dt \\
&\quad - (\beta - 1) r L_t dt + \beta \frac{L_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) \sigma_R S_t dW_t \\
&= \beta \frac{L_t}{V(t,S_t)} \Bigg(rV(t,S_t) + \frac{1}{2}(\sigma_R^2 - \sigma_I^2) S_t^2 \frac{\partial^2 V}{\partial s^2}(t,S_t) \\
&\quad + (\mu - r) S_t \frac{\partial V}{\partial s}(t,S_t) \Bigg) dt - (\beta - 1) r L_t dt + \beta \frac{L_t}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) \sigma_R S_t dW_t
\end{aligned}
$$

Hence (denote with $\tilde{L}$ the delta-hedge + short proceedings strategy)

$$
\begin{aligned}
d\tilde{L}_t &= dL_t - \beta L_t \frac{1}{V(t,S_t)} \frac{\partial V}{\partial S}(t,S_t) dS_t + rS_t \beta L_t \frac{1}{V(t,S_t)} \frac{\partial V}{\partial s}(t,S_t) dt \\
&= \beta \frac{L_t}{V(t,S_t)} \left(rV(t,S_t) + \frac{1}{2}(\sigma_R^2 - \sigma_I^2) S_t^2 \frac{\partial^2 V}{\partial s^2}(t,S_t) \right) dt \\
&\quad - (\beta - 1) r L_t dt \\
&= rL_t dt + \beta \frac{L_t}{V(t,S_t)} \frac{1}{2}(\sigma_R^2 - \sigma_I^2) S_t^2 \frac{\partial^2 V}{\partial s^2}(t,S_t) dt
\end{aligned}
$$

So the Delta-hedging strategy of the LETF-like portfolio scales the dollar Gamma exposure $S_t^2 \frac{\partial^2 V}{\partial s^2}(t,S_t)$ by the (local) leverage multiplier $\beta L_t / V(t,S_t)$, under the above model relaxation.

References

[1] Avellaneda, M. and Zhang, S. (2010). *Path-Dependence of Leveraged ETF Returns*. SIAM Journal on Financial Mathematics 1(1), 586–603.