The mathematics of α-quantile options: an introduction

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General definition of the α-quantile

Let $X = (X_t)_{t \geq 0}$ be a stochastic process; this can be multidimensional - e.g. a basket of equities. We fix the following parameters:

- A **time of maturity** $T > 0$ at which we will evaluate the payoff;
- A **constant vector** γ **in** $\mathbb{R}^d$ - with the same dimension as the number of assets in the price process;
- The **quantile level** $\alpha \in (0, 1)$.

We thus define the *hyperplane* α-quantile on X with the given parameters (T, γ, α) as:

$$M_{T,\alpha}(X) = \inf \left\{ y : \frac{1}{T} \int_0^T \mathbf{1}_{\{z : \gamma \cdot z \leq y\}}(X_s) ds \geq \alpha \right\} \tag{1}$$

That is: equation (1) is the smallest real value y such that the asset price process X has passed a proportion α of the total time up to maturity T in the (closed) *lower half-space* region $\{z \in \mathbb{R}^d : \gamma \cdot z \leq y\}$.

An example on a Black-Scholes log-price path

Let us consider the example of the Black-Scholes model for a single asset S under the risk-neutral dynamics with rate $r = \sigma^2/2$ for simplicity:

$$dX_t = \frac{\sigma^2}{2} X_t dt + \sigma X_t dW_t^Q, \quad t \leq T$$

so that $d \ln X_t = \sigma dW_t^Q$. The **distribution** of the α-quantile of the log-price process $\ln(X_\cdot / X_0)$ is known (see the references) and we can derive the following **closed-form result** for the Q-expectation of the α-quantile with $\gamma = 1$:

$$E^Q \left[M_{T,\alpha} \left(\ln \frac{X_\cdot}{X_0} \right) \right] = \sigma \frac{\sqrt{2\alpha T} - \sqrt{2(1-\alpha)T}}{\sqrt{\pi}} \tag{2}$$

In particular, equation (2):

- Expresses the average smallest value under which the log-price passes at least αT time in *absence of arbitrage*. The equivalent expectation under the **physical measure** P would be a **risk management** tool.
- Depends *linearly* on the volatility parameter $\sigma > 0$ and *nonlinearly* on the quantile level α (and the maturity time T) - as is shown in Figure 1.

A simulation of a log-price path and two corresponding α-quantiles for different levels are shown in Figure 2.

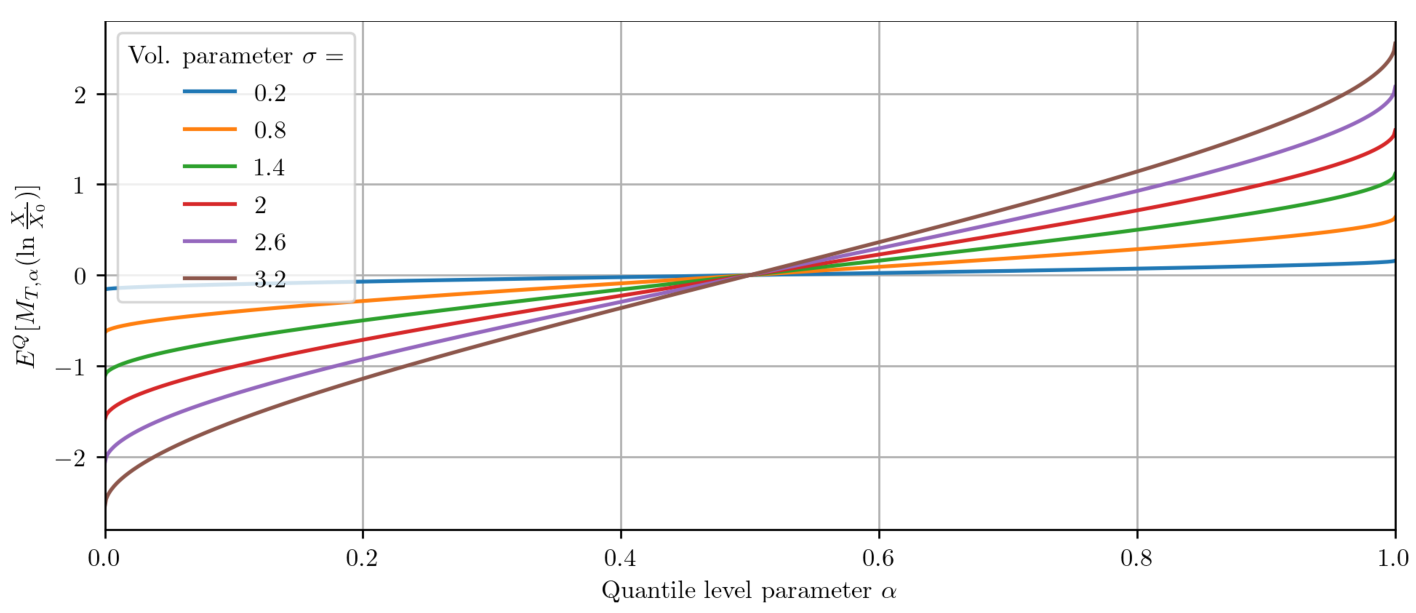

Figure 1. *Q-expectation of the α-quantile (eq. (2) on the Black-Scholes log-price for $T = 1$.*

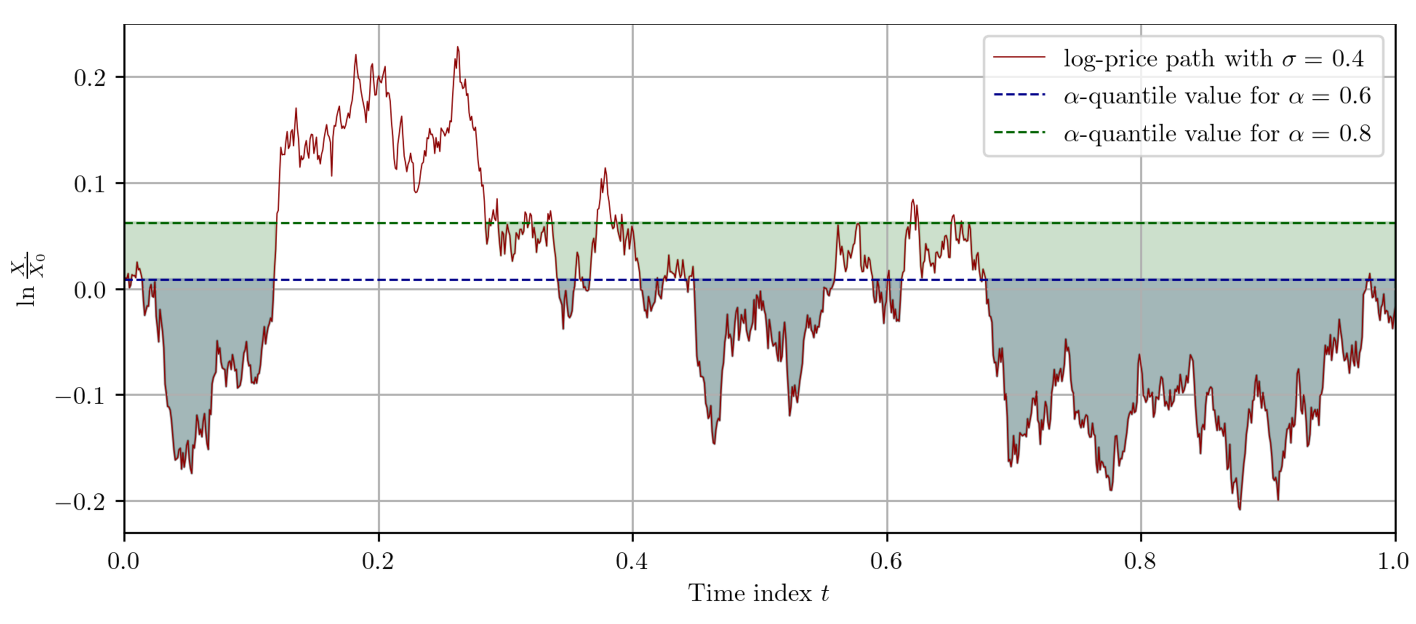

Figure 2. *α-quantiles of a risk-neutral path of the Black-Scholes log-price for $T = 1$.*

Mathematical properties of the α-quantile

The α-quantile of a process X as defined in equation (1) can be considered as a **path functional** acting on the space of $\mathbb{R}^d$-valued càdlàg functions, which is called the *Skorokhod space*: $\mathcal{D}_{\mathbb{R}^d}[0, \infty) \ni x \mapsto M_{T,\alpha}(x)$.

This is because we generally consider asset price models which have at least càdlàg paths, that is $X : \Omega \to \mathcal{D}_{\mathbb{R}^d}[0, \infty)$, e.g. **jump-diffusions** or more general **semimartingales**, or just diffusions like the Black-Scholes model which have continuous paths.

In particular, the hyperplane α-quantile path functional:

- It is a measurable function, and for any $T > 0$ and $x \in \mathcal{D}_{\mathbb{R}^d}[0, \infty)$ fixed it is **nondecreasing** and **left-continuous** *in the argument* α, with at most countable discontinuities;
- For a quantile level α fixed, the functional is continuous over the Skorokhod space at all x s.t. $\alpha \notin \{\alpha : M_{T,\alpha}(x) < M_{T,\alpha^+}(x)\}$ - that is we have an **explicit continuity set** over its domain.

From this we can prove the following theorem.

Convergence in distribution of the α-quantile

We may want to know if - for example - given a sequence of **simulation schemes** $(X^n)_{n \in \mathbb{N}}$ that converges to the *true* asset price process X (in the sense of distributions) we also have the convergence of the α-quantile.

If, alternatively:

- For any $\varepsilon > 0$ we have $P(M_{T,\beta}(X) > M_{T,\alpha}(X) + \varepsilon) \to 0$ as $\beta \downarrow \alpha$;
- X has a.s. continuous paths;

then $M_{T,\alpha}(X^n) \to^d M_{T,\alpha}(X)$. This is a *continuous mapping* theorem.

The first hitting time of $M_{T,\alpha}(X)$

In fact, the following is also true:

- We can define well the **first hitting time** - the time at which X has hit its own α-quantile before time T (*it is a nontrivial example of a random time which is not a stopping time!*) and in the particular case of a $\mathbb{R}$-valued continuous-path process X we denote it with

$$\tau_{M_{T,\alpha}}(X) = \inf\{t \leq T : X_t = M_{T,\alpha}(X)\}$$

- We can then consider the α-quantile and this random time jointly and we can find an explicit **joint continuity set**.

By using deep results of the properties of Brownian motion, we can prove that if X^n converges to a (scaled) $\mathbb{R}^d$-Brownian motion, then the α-quantile and its random time converge **jointly in distribution**.

References: some literature review on the topic

- The α-quantile can be used to construct **path-dependent option payoffs** - that is *exotic derivatives* with α-quantile underlying. The no-arbitrage pricing of derivatives such as α-quantile call options

$$e^{-r(T-t)} E^Q[(Y_0 e^{M_{T,\alpha}(X)} - K)^+ | \mathscr{F}_t], \quad Y_0 > 0$$

is explored e.g. in **(Dassios, A.; *Ann. Appl. Probab.*, 1995)** by also deriving explicitly the Brownian distribution of the α-quantile; see also **(Yor, M.; *J. Appl. Probab.*, 1995)** and **(Akahori, J.; *Ann. Appl. Probab.*, 1995)**
- The joint density of the Brownian α-quantile and its first and last hitting times is derived in **(Dassios, A.; *Bernoulli*, 2005)**; this result can be used to price exotic options depending jointly on all these random quantities.
- The introduction of the α-quantile of stochastic processes dates back to **(Miura, R.; *Hitotsubashi Journal of Commerce and Management*, 1992)**.
- Results on the α-quantiles of processes with exchangeable increments are explored in **(Chaumont, L.; *J. Lond. Math. Soc.*, 1999)**.